Fourier Analysis

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Review

- Convolutions.

Let f, g be integrable on the circle,

$$f*g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot g(x-y)\,dy, \quad x\in[-\pi,\pi].$$

- $f*g$ is cts on the circle
- $\widehat{f*g}(n) = \hat{f}(n)\,\hat{g}(n), \quad n\in\mathbb{Z}.$

Example 1. Let $e_n(x) := e^{inx}$.

Then for every integrable f on the circle

$$e_n * f(x) = \hat{f}(n)\,e^{inx}.$$

Check:
$$e_n * f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(x-y)} f(y)\,dy$$
$$= e^{inx}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,e^{-iny}\,dy$$
$$= \hat{f}(n)\,e^{inx}.$$

Example 2. Let $g(x) = \sum_{n=-N}^{N} c_n e^{inx}$.

Then
$$g * f(x) = \sum_{n=-N}^{N} c_n \cdot \left(e_n * f(x)\right)$$
$$= \sum_{n=-N}^{N} c_n \hat{f}(n) e^{inx}.$$

In particular, letting $g(x) = D_N(x) = \sum_{n=-N}^{N} e^{inx}$ (the N-th Dirichlet kernel),

we have
$$g * f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = S_N f(x).$$

§2.4. A Convergence Thm for "good kernels".

Def. A good kernel on the circle is a sequence $\{K_n\}_{n=1}^{\infty}$ of integrable functions on the circle satisfying the following 3 properties:

(1) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1 \quad \text{for all } n \in \mathbb{N}.$$

(2) $\exists$ a constant $M > 0$ such that

$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \le M \quad \text{for all } n \in \mathbb{N}$$

(3) $\forall\ 0 < \delta < \pi$,

$$\int_{\delta < |x| < \pi} |K_n(x)|\,dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thm 1. Let $\{K_n\}_{n=1}^{\infty}$ be a good kernel on the circle. Let f be integrable on the circle. Then

$$\lim_{n\to\infty} K_n * f(x) = f(x)$$

provided f is cts at x. Moreover if f is cts on the circle, then

$$K_n * f(x) \rightrightarrows f(x) \quad \text{on the circle}$$

as $n \to \infty$.

Proof. Suppose f is cts at x.

Let $\varepsilon > 0$. Then we can pick $\delta > 0$ such that

$$|f(x-y) - f(x)| < \varepsilon \quad \text{for all } |y| < \delta.$$

Notice that

$$K_n * f(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) f(x-y)\,dx$$

$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y) \cdot f(x)\,dy$$

(here we used $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = 1$)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\big(f(x-y) - f(x)\big)\,dy$$

Hence

$$|K_n * f(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_n(y)| \cdot |f(x-y) - f(x)|\,dy$$

$$= \frac{1}{2\pi}\int_{\delta < |y| < \pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy$$

$$+ \frac{1}{2\pi}\int_{|y| \leq \delta} |K_n(y)| \cdot |f(x-y) - f(x)|\,dy$$

$$\leq \frac{B}{\pi}\int_{\delta < |y| < \pi} |K_n(y)|\,dy + \frac{\varepsilon}{2\pi}\int_{|y| \leq \delta} |K_n(y)|\,dy$$

($B := \sup |f|$)

$$\leq \frac{B}{\pi} \int_{\delta < |y| < \pi} |K_n(y)| dy + \frac{\varepsilon}{2\pi} \cdot M$$

$$\leq 2 \times \left(\frac{\varepsilon}{2\pi} \cdot M \right)$$ when n is large enough

Hence $K_n * f(x) \to f(x)$ as $n \to \infty$.

By modifying the above argument slightly, we can show that if f is cts on the circle, then the above convergence is uniform on the circle. ▨

- An example of good kernels —— Fejer kernel

Set for $N = 1, 2, 3, 4, \cdots$

$$F_N(x) := \frac{1}{N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}, \quad x \in [-\pi, \pi].$$

We call $(F_N)_{N=1}^{\infty}$ the Fejer Kernel.

- $$F_N(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N}$$

$$\left(\text{where} \quad D_k(x) = \sum_{|n| \le k} e^{inx} \right)$$

$$= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{inx}.$$

- $(F_N)_{n=1}^{\infty}$ is a good kernel.

Check: ① $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x)\,dx$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{D_0(x) + \cdots + D_{N-1}(x)}{N} dx$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_k(x)\,dx$$

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases} \right)$$

$$= 1$$

② Since $F_N(x) \geq 0$,

$$\int_{-\pi}^{\pi} |F_N(x)|\,dx = 2\pi \quad \text{for all } n.$$

③ For given $0<\delta<\pi$, if $\delta<|x|<\pi$,

$$F_N(x) = \frac{1}{N} \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}$$

$$\leq \frac{1}{N} \frac{1}{\sin^2 \frac{\delta}{2}} \to 0 \quad \text{as } N\to\infty$$

So

$$\int_{\delta<|x|<\pi} |F_N(x)|\,dx \leq 2\pi \cdot \frac{1}{N} \cdot \frac{1}{\sin^2 \frac{\delta}{2}}$$

$$\to 0 \quad \text{as } N\to\infty.$$